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Some Estimations of Tolerance Bands of S-N Curves

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A precise determination of tolerance bands of S-N curves is quite difficult problem; therefore simple approximate constructions of these bands are often used. Castillo et al. suggested a sophisticated and at the same time easy procedure for their estimation which, however, needs a special type of regression function for describing S-N curve. This type of regression function called the Castillo function is suitable only for high-cycle fatigue region close to fatigue limit. The paper focuses on other regression functions used in the Castillo procedure, namely on the Kohout and Vechet function and the Stromeyer function.
Keywords: S-N curve, tolerance band, Weibull distribution; regression, maximum likelihood method.

1. INTRODUCTION

Having the results of fatigue tests and choosing a suitable regression function, it is quite easy to determine so-called mean regression curve which divides the set of experimental points into two more or less equally numerous subsets lying below and above that curve. The construction of tolerance bands specifying other fraction of the results lying below the curve than 50 % is quite complicated mainly due to substantially varying dispersion of results along the regression curve (this property is called heteroskedasticity). Just it is the most cogent argument why the number of cycles to fracture (or the logarithm of this number) is better to consider as independent variable and the stress amplitude as dependent variable even if in reality it is just vice versa. This replacement can substantially decrease the level of heteroskedasticity so that mostly it needs not be taken into account.

For the construction of tolerance bands it is necessary to know the type of statistical distribution of experimental results. In the case of fatigue the best description can be obtained using the three-parameter Weibull distribution. In this case the accurate construction of tolerance bands is particularly complicated if the problem is not solved by a professional statistician. Hence sometimes approximate constructions are proposed which are sufficiently simple and simultaneously they give sufficiently usable results. On the other hand, to use approximate methods can mean a loss of some specific features of accurately constructed tolerance bands, in presented case it is usual shape of tolerance bands which are the narrowest in the middle of regression curve and they extend towards its ends.

2. CASTILLO PROCEDURE

The Castillo approach to the approximate construction of tolerance bands [1 – 3] is based on regression function

$$\log\frac{N}{N_0}\log\frac{\sigma}{\sigma_\infty} = A \qquad (1)$$

describing the dependence between the fatigue stress (upper stress or stress amplitude of loading cycle) σ and the number of cycles to fracture N. Among three regression parameters A, N_0 and σ_∞ the two following have clear meaning: σ_∞ is permanent fatigue limit (i. e. the limit for infinite number of cycles to fracture) and N_0 is the number of cycles for which the stress is infinitely high. Function (1) can be meaningfully used only if $\sigma > \sigma_\infty$ and $N > N_0$. While the limitation from below by permanent fatigue limit is very natural, the limitation by certain number of cycles to fracture (even if it can be very low) represents quite substantial deficiency of the Castillo function in comparison with e. g. the Stromeyer function [4] because the condition $N > N_0$ limits the Castillo function applicability only for high-cycle region.

The application of the Castillo approach [1 – 3] is quite simple: regression function

$$\log N = \frac{A}{\log\dfrac{\sigma}{\sigma_\infty}} + \log N_0 \qquad (2)$$

easily derived from Eq. (1) is used for the determination of regression parameters N_0 and σ_∞. Then the set of values

$$A_i = \log\frac{N_i}{N_0}\log\frac{\sigma_i}{\sigma_\infty} \qquad (3)$$

corresponding to experimental data couples (N_i, σ_i) is calculated which is assumed to have the three-parameter Weibull distribution. For the determination of the parameters of this distribution the maximum likelihood method is used. Tolerance limits are then calculated using Eq. (2) where corresponding fractiles (quantiles) of the distribution of A values are introduced.

The main advantage of the Castillo approach consists in the fact that standard deviations of regression parameters are not necessary. It means that regression (minimization of the sum of deviation squares) as well as the determination of the parameters of the Weibull distribution (maximization of likelihood) can be simply done in MS Excel program using *Solver* supplement, only its *default options* must be suitably changed in order to reliably solve both the extremizations with high degree of

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nonlinearity [5]. However the Castillo approach is principally simple, its results are very good estimation of accurately determined tolerance bands and they are very useful above all when the sets of the results of various fatigue tests (e. g. for various materials or different asymmetries of loading cycle) are compared.

Besides the regression function (2) applied in the Castillo approach also the inverse regression function

$$\sigma = \sigma_\infty 10^{\frac{A}{\log \frac{N}{N_0}}} \quad \text{or} \quad \sigma = \sigma_\infty \exp \frac{A}{\ln \frac{N}{N_0}} \tag{4}$$

can be principally used (both the forms of Eq. (4) are equivalent) as it will be shown later.

3. DETERMINATION OF PARAMETERS AND FRACTILES OF WEIBULL DISTRIBUTION

The distribution function of the three-parameter Weibull distribution is usually written in the form

$$\begin{aligned} P(x) &= 1 - \exp\left[-\left(\frac{x-c}{a}\right)^b\right] && \text{for} \quad x \geq 0, \\ P(x) &= 0 && \text{for} \quad x \leq 0 \end{aligned} \tag{5}$$

where a is so-called measure parameter, b is shape parameter and c is position parameter (for $c = 0$ the two-parameter Weibull distribution is obtained). For the determination of distribution parameters various methods can be used, in this case in accordance with the Castillo approach the maximum likelihood method will be applied: for final values of distribution parameters a, b and c the expression

$$\ln L(a,b,c) = n \ln b - n\, b \ln a + (b-1)\sum_{i=1}^{n} \ln(x_i - c) - \frac{1}{a^b}\sum_{i=1}^{n}(x_i - c)^b \tag{6}$$

reaches its maximum [6]. The fractiles of the distribution can be determined using the relation

$$x_p = c + a\,[-\ln(1-p)]^{1/b} \quad \text{for} \quad 0 \leq p \leq 1 \tag{7}$$

where $100\,p$ % is the probability corresponding to the mentioned fractile.

4. THE APPLICABILITY OF THE KOHOUT AND VECHET FUNCTION

Fortunately also the Kohout and Vechet function describing S-N curves [7 – 9]

$$\sigma = \sigma_\infty \left(\frac{N}{N+C}\right)^b \tag{8}$$

can be rewritten into the form

$$\sigma \left(\frac{N+C}{N}\right)^b = \sigma_\infty \tag{9}$$

principally similar to Eq. (1) i. e. in common general form $f_1(N)\cdot f_2(\sigma) = A$. The main advantages of the Kohout and Vechet function in comparison with the Castillo function are two: (i) no limiting condition of $N > N_0$ type, (ii) instead of the distribution of an *obscure* parameter A directly the distribution of permanent fatigue limit σ_∞ is calculated.

5. COMPARISON OF THE APPLICATION OF DIFFERENT REGRESSION FUNCTIONS

For comparison of the Castillo function with the Kohout and Vechet function in the Castillo approach the results of Zapletal et al. published in paper [10] are used, excluding the values of ultimate tensile strength. The resulting tolerance bands depend on that which of regression functions (2), (4) or (8) is used for regression. The application of regression function (2) leads to the original Castello procedure with the results given in Fig. 1. In this figure as well as in all others the fractiles 0 % (with high probability no result appears below this curve), 5 %, 50 % and 95 % are presented (the curve for 100 % lies in infinity). The curve for 50 % should be principally identical with the mean regression curve only in the case that the distribution of experimental results is symmetrical. The Weibull distribution is symmetrical for no values of its parameters but in all studied cases its asymmetry is not high enough for graphical resolution between mean regression curve and 50 %-curve.

As already mentioned, the application of stress amplitude as dependent variable leads due to high degree of heteroskedasticity of log$N$ values to more *natural* regression results. It is documented in Fig. 2 where regression function (4) is used.

The application of the Kohout and Vechet regression function (8) is presented in Fig. 3. While in this figure with linear stress scale the tolerance bands dilate towards low-cycle region, in Fig. 4 with logarithmic stress scale the curves defining these bands are approximately parallel or, strictly speaking, all the curves have the same shape and they are only shifted in the direction of stress axis (it is general property of all regression functions which can be written in the special form $f(N)\cdot\sigma = A$).

6. DISCUSSION

Comparing the curves defining the tolerance bands in the Castillo procedure using the original Castillo regression function (2) (see Fig. 1) with the curves corresponding to inverse Castillo function (4) (see Fig. 2), it is possible to say that in the second case the mean regression curve as well as the tolerance bands correspond better to intuitive idea how they should look like. Besides this purely subjective criterion following objective criteria on behalf of the inverse function can be listed:

a) leads to narrower tolerance bands (above all in low-cycle region),
b) gives better fit in the region of fatigue limit which is usually most important result of fatigue tests,
c) reduces substantially or nearly removes the influence of heteroskedasticity of the set of fatigue results (as already mentioned above).

Replacing the inverse Castillo function by the Kohout and Vechet function in the modified Castillo procedure, the following findings can be enunciated:

a) even narrower tolerance bands are obtained,
b) more suitable description of high-cycle region is obtained, above all the region of fatigue limit,

which is evidence that the Kohout and Vechet function (at least in presented case but many other examples exist) describes the set of fatigue test results substantially better than the Castillo function.

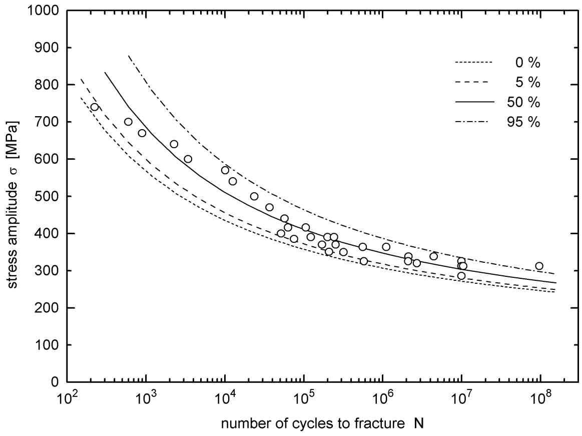

Fig. 1. Mean regression curve identical with the curve for 50 % percentile and the curves for presented percentiles according to the original Castillo procedure based on the Castillo regression function (2)

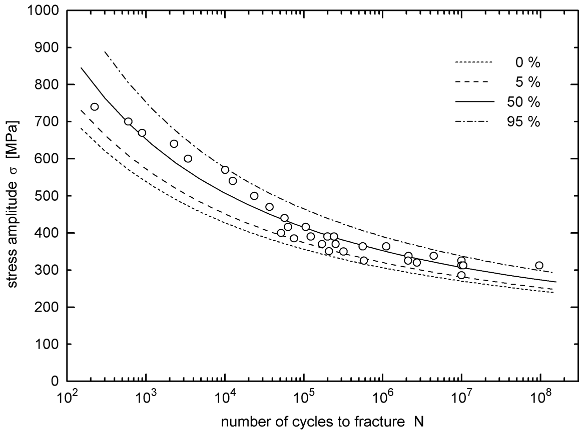

Fig. 2. Mean regression curve identical with the curve for 50 % percentile and the curves for presented percentiles according to the modified Castillo procedure based on the inverse Castillo regression function (4)

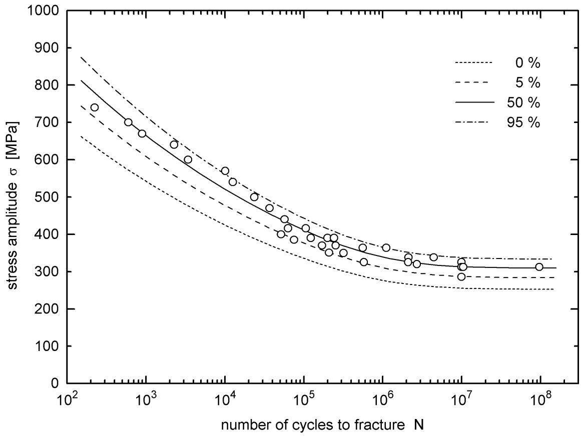

Fig. 3. Mean regression curve identical with the curve for 50 % percentile and the curves for presented percentiles according to the modified Castillo procedure based on the Kohout and Vechet regression function (8)

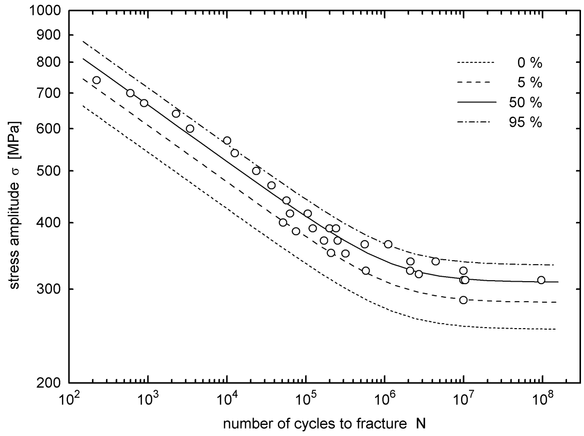

Fig. 4. Mean regression curve identical with the curve for 50 % percentile and the curves for presented percentiles according to the modified Castillo procedure based on the Kohout and Vechet regression function (8) in the case of logarithmic stress scale

Fig. 4 presents two following facts:

a) in the modified Castillo procedure with the Kohout and Vechet regression function (and all other functions writable in the form $f(N)\cdot\sigma = A$) the curves corresponding to different percentiles differ only in multiple (if logarithmic stress scale is used, they are only shifted in the direction of stress axis),

b) the Kohout and Vechet function is presented in $\log N - \log\sigma$ fit by two half-lines connected by relatively sharp bend.

Just the curve presented (approximately) by two half-lines (oblique and horizontal) best correspond to classical representation of S-N curves.

Application of the three-parameter Weibull distribution has cardinal importance in the Castillo approach. Using only the two-parameter Weibull distribution the 0 % fractile would be trivial: $\sigma = 0$. Hardly anywhere the absurdity of the application of the two-parameter Weibull distribution is such clear as in this case. If extremely brittle materials with extremely low mechanical properties presenting extremely high dispersion are not studied (neither structural ceramics have such properties in this time) it is better to avoid the application of this distribution.

For using the Kohout and Vechet function in the Castillo procedure the close similarity between Eqs (1) and (9) has been exploited. On the other hand, more natural approach to approximate construction of tolerance bands is to study the three-parameter Weibull distribution of the differences $\Delta\sigma_i$ between measured and fitted values than e. g. the distribution of the value σ_∞. The results based on $\Delta\sigma$ distribution are presented in Fig. 5. With respect to the principle of tolerance bands this construction is the most rightful and namely unambiguous because in principle not only the regression parameters A in the Castillo function or σ_∞ in the Kohout and Vechet function, but all other regression parameters in each of the function can be in principle used as the basic one whose distribution is determined. Different basic parameters mean different tolerance bands and which of them are the correct ones? This ambiguity represents the weakest point of the Castillo approach.

The results based on $\Delta\sigma$ distribution are presented for the Kohout and Vechet function in Fig. 5. This approach is applicable for all regression functions including very frequently used the Stromeyer function [4]

$$\sigma = aN^{b} + \sigma_{\infty}, \tag{10}$$

whose similarity with the Castillo function is not so close than the similarity of the Kohout and Vechet function (e. g. it cannot be rewritten in the common form $f_1(N)\cdot f_2(\sigma) = A$), see Fig. 6. Comparison of both last figures unambiguously shows the earlier published fact [7 – 10] that the Kohout and Vechet function allows better fit of S-N curves with respect to the Stromeyer function, especially in the region of fatigue limit.

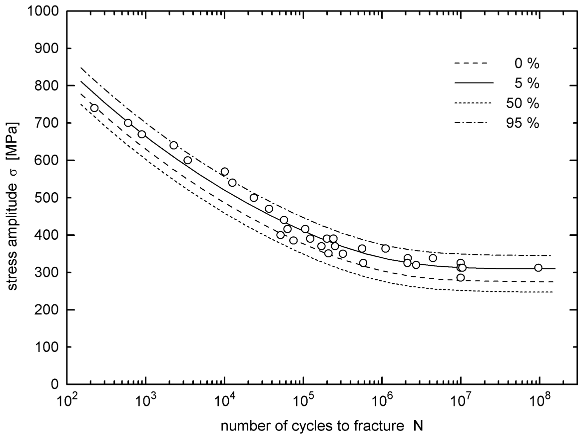

Fig. 5. Mean regression curve identical with the curve for 50 % percentile and the curves for presented percentiles based on the Kohout and Vechet regression function (8) and on the distribution of differences between measured and fitted values

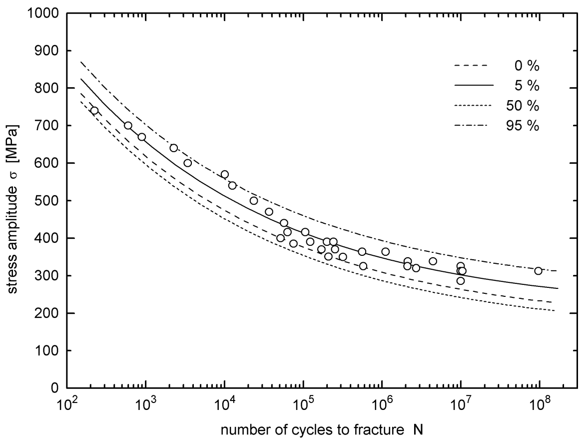

Fig. 6. Mean regression curve identical with the curve for 50 % percentile and the curves for presented percentiles based on the Stromeyer regression function (10) and on the distribution of differences between measured and fitted values

7. CONCLUSIONS

1. The Castillo procedure enables approximate but very useful construction of tolerance bands of S-N curves. It consists in the nonlinear regression of fatigue test results using the Castillo regression function and in the determination of the three-parameter Weibull distribution using the maximum likelihood method. As it does not need the standard deviation of regression parameters, it can be easily performed using MS Excel which is accessible practically on every PC.
2. Using the inverse Castillo regression function instead of the origin one in the Castillo procedure, i. e. $\sigma = f(N)$, the more favourable results (above all the narrower tolerance bands) are obtained in comparison with the original Castillo procedure.
3. Even better results are obtained with using the Kohout and Vechet regression function in the Castillo procedure.
4. The most natural and unambiguous approach to approximate construction of tolerance bands is obtained if it is based on the three-parameter Weibull distribution of the differences between measured and fitted values. Using this approach for the Kohout and Vechet regression function and for the Stromeyer function, the comparison of resulting constructions speaks in favour of the Kohout and Vechet function.

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