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Numerical Method for Analysis of Waveguide Modes in Planar Gradient Waveguides

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This paper presents a numerical method developed to find propagation constants of planar waveguide localized modes. The method is based on both the Fourier transform application and the wave equation solution in a frequency domain. As a result, integral equation is obtained where integral is replaced by sum. Finally, a task to find propagation constants and field Fourier transforms in a discrete form is led to the eigenvalue/eigenvector problem. This method provides high accuracy subject to the conditions of the Whittaker-Shannon sampling theorem, and it is characterized by high numerical stability. The method is tested on many examples.
Keywords: Fourier transform, convolution, propagation constant, spatial frequency, wave equation.

1. INTRODUCTION

Planar waveguides are widely used in many modern optoelectronic devices, such as semiconductor lasers [1], distributed feedback lasers [2, 3], various devices of integrated optics [4]. Moreover, a planar waveguide may consist of many layers, and a permittivity can be changed according to certain function on coordinate – it is so-called gradient waveguides. If permittivity is constant within each layer, transcendental dispersion equation is obtained which can be solved by numerical method [5, 6]. However, this method is too cumbersome, and one has to write new transcendental equation when number of layers increases. In [5 – 7], analytical solutions for some planar gradient waveguides are presented with the usage of methods developed in quantum mechanics to solve one-dimensional stationary Schrödinger equation [8]. Besides, to find propagation constants and appropriate electromagnetic fields of gradient waveguide modes the WKB-approximation technique is used [9], which is also taken from quantum mechanics. For finding propagation constants direct numerical solutions of Maxwell's equation [9] and many other methods are used: a method formulated using the four-sheeted Riemann surface of the analytic function that defines the waveguide dispersion relation [10]; transfer matrix method [11]; matrix approach [12], where matrix multiplication is used. Propagation constants of waveguide modes can be also determined with the resonance phenomena [13, 14].

For analysis of multilayer waveguides a numerical method is proposed [15, 16], where in wave differential equation the second derivative of field is replaced by appropriate difference operator. Writing difference equation for many coordinates, we obtain well-known eigenvalue/ /eigenvector problem of matrix algebra. Eigenvalues obtained correspond to waveguide mode propagation constants and eigenvectors – to field distribution in waveguide. This method can also be applied to the gradient waveguides. However, it is effective only for TE polarization waves, whose wave equation contains only the second derivative of coordinate. For TM polarization waves in wave equation the first derivative of permittivity is present. Thus, for planar waveguides in which permittivity is changed of the coordinate skipping, the first derivative tends to infinity, making this method difficult to usage. Perhaps, for this reason a numerical analysis in [15, 16] was carried out only for TE polarization modes. It is obviously, if permittivity is a continuous function, this method is also suitable for TM polarization waves. However, accuracy of analysis can be reduced due to the presence of the first derivative of field [5]. In this method, differential equation is replaced by difference one. For achievement of high accuracy it is necessary to take a small step along coordinate axis. As a result, in order to determine propagation constants one needs to search eigenvalues of large size matrix. However, in was shown according to numerical experiments that error increases in this case as numerical differentiation is a source of noise due to rounding in a numerical process. The results of [16] show low accuracy of this method.

Wave equation for planar waveguides of TE polarization waves [5] is identical to one-dimensional stationary Schrödinger equation [8]. For this reason, to determine propagation constants of gradient planar waveguide modes many approximate methods are used, for example, those which were initially developed for problems of quantum mechanics [9]. It should be noted, that even at present a search of problems in quantum mechanics with exact analytical solutions continues [17 – 19]. Therefore, an effective numerical method for solving the Schrödinger equation can help to find and verify exact analytical solutions. A numerical method can be also used to verify the approximate methods in quantum mechanics. That's why, the numerical method of its high accuracy and simplicity could find applications in waveguide technology and quantum mechanics. The current state of computer technology and software sophistication allows applying numerical

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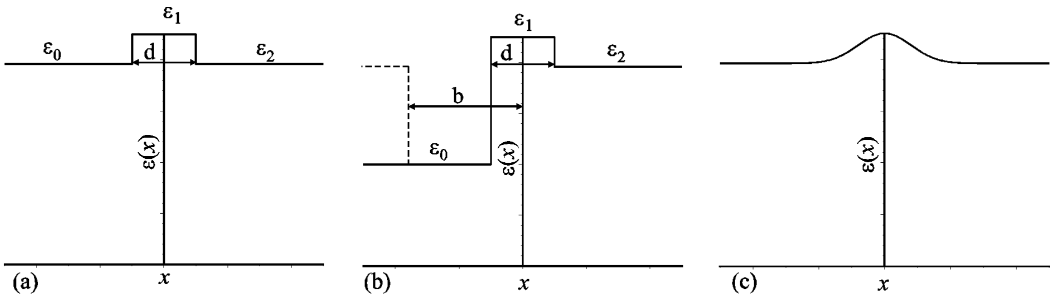

Fig. 1. The simplest planar waveguides: symmetric planar waveguide $\varepsilon_0 = \varepsilon_2 < \varepsilon_1$ (a); asymmetric planar waveguide $\varepsilon_0 < \varepsilon_2 < \varepsilon_1$ (b); symmetric gradient waveguide $\varepsilon(-\infty) = \varepsilon(\infty) < \varepsilon(0)$ (c)

methods to find propagation constants and field distributions of gradient planar waveguides. Samples of the simplest planar waveguides are presented in Fig. 1.

It should be noted that known methods to search waveguide mode propagation constants (discrete energy levels) are based on wave equation solution in coordinate area. For localized waveguide modes field intensity and the first derivatives of coordinate x in $\pm\infty$ are zero. Thus, for field distribution the Fourier transform exists [20], and appropriate wave equation can be converted into frequency domain using the Fourier transform. As a result, we obtain integral equation which can also be solved by numerical methods.

The purpose of this study is to develop a new numerical method to solve one-dimensional wave equation for planar waveguides using the Fourier transform and to demonstrate some of its advantages in comparison with the methods exist.

In section 2 the essence of the method proposed is described. In section 3 the results of numerical analysis of planar waveguides for TE and TM polarization waves are presented.

2. ONE-DIMENSIONAL WAVE EQUATIONS AND THEIR FOURIER TRANSFORMS

A structure of the simplest planar symmetric waveguide is shown in Fig. 2. If, in this waveguide $\varepsilon_1 > \varepsilon_0$, propagation of localized waveguide mode of propagation constant β is possible. Moreover, electric field intensity of electromagnetic wave can be described as:

$$E(x, z) = E(x)\exp(-i\beta z). \quad (1)$$

However, even in this simplest case a search of propagation constant is reduced to solution of transcendental algebraic equation [5]. The problem becomes more difficult if permittivity is changed according to complex function along coordinate axis x.

If, in a waveguide mode, electric field intensity is perpendicular to plane xz (TE polarization), wave equation has following form [21]:

$$\frac{d^2E(x)}{dx^2} + \left(\frac{2\pi}{\lambda}\right)^2 \varepsilon(x)E(x) = \beta^2 E(x), \quad (2)$$

where λ is wavelength.

If, in a waveguide, TM polarization wave propagates, appropriate wave equation according to magnetic field intensity can be written as:

$$\frac{d^2H(x)}{dx^2} - \frac{1}{\varepsilon(x)}\frac{d\varepsilon(x)}{dx}\frac{dH}{dx} + \left(\frac{2\pi}{\lambda}\right)^2 \varepsilon(x)H(x) = \beta^2 H(x). \quad (3)$$

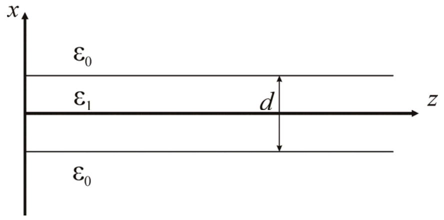

Fig. 2. Symmetric planar waveguide of two fixed values of permittivity

Functions $E(x)$, $H(x)$ that describe fields in localized waveguide modes and their first derivatives tend towards zero if $x \to \pm\infty$. That's why, for these functions, their first and second derivatives the Fourier transform exists. One can write appropriate equations for $E(x)$. Therefore, Fourier transforms for $E(x)$, first and second derivatives of $E(x)$ are [20]:

$$F\{E(x)\} = E(u) = \int_{-\infty}^{\infty} E(x)\exp(-i2\pi ux)dx, \quad (4)$$

$$F\left\{\frac{dE(x)}{dx}\right\} = i2\pi uE(u) = \int_{-\infty}^{\infty} \frac{dE(x)}{dx}\exp(-i2\pi ux)dx, \quad (5)$$

$$F\left\{\frac{d^2E(x)}{dx^2}\right\} = -(2\pi u)^2 E(u) = \int_{-\infty}^{\infty} \frac{d^2E(x)}{dx^2}\exp(-i2\pi ux)dx, \quad (6)$$

where u is the spatial frequency.

Besides, for functions for which Fourier transforms exist, i. e., $F\{g(x)\} = G(u)$, $F\{h(x)\} = H(u)$, next equation is yet right:

$$F\{g(x)h(x)\} = \int_{-\infty}^{\infty} G(u-v)H(v)dv, \quad (7)$$

where $F\{...\}$ is the Fourier transform, v is spatial frequency too. Equation (7) expresses the essence of the convolution theorem [20].

Let's consider Fourier transforms of left and right parts of (2, 3), taking into account (4 – 7). As a result we obtain:

$$-4\pi^2u^2E(u) + \left(\frac{2\pi}{\lambda}\right)^2 \int_{-\infty}^{\infty} \varepsilon(u-v)E(v)dv = \beta^2E(u), \quad (8)$$

$$-4\pi^2u^2H(u) - 2i\pi \int_{-\infty}^{\infty} F\left\{\frac{1}{\varepsilon(x)}\frac{d\varepsilon(x)}{dx}\right\}(u-v)vH(v)dv + \left(\frac{2\pi}{\lambda}\right)^2 \int_{-\infty}^{\infty} \varepsilon(u-v)H(v)dv = \beta^2H(u). \quad (9)$$

Thus, we have moved from differential equations (2, 3) to integral ones (8, 9). In these last equations we can replace integral by sum. For example, if we take (8), resulting in a replacement of continuous values u and v by discrete ones we obtain:

$$-4\pi^2(u_s)^2E(u_s)+\left(\frac{2\pi}{\lambda}\right)^2\sum_{k=-(N-1)/2}^{(N-1)/2}\varepsilon(u_s-v_k)E(u_s)\Delta =$$

$$=\beta^2E(u_s), \quad (10)$$

where $\Delta = u_{max}/N$, $u_s = s\Delta$, $v_k = k\Delta$; $-(N-1)/2 \le s, k \le (N-1)/2$. u_{max} is a value of interval in a frequency domain $-u_{max}/2 \le u \le u_{max}/2$; N is a set of points, in which function $E(u_s)$ is sought. Beyond this interval it is assumed that $E(u_s) = 0$. A value of N should be taken large enough and unpaired.

Let's write (10) for all discrete frequencies $u_s = s\Delta$. Moreover, s is changing from $-(N-1)/2$ to $(N-1)/2$. Then a set of these equations will be written as matrix equation while β^2 is common to all values of s:

$$(\mathbf{P}+\mathbf{U})\mathbf{E}=\beta^2\mathbf{E}, \quad (11)$$

where $\mathbf{P}$ is the diagonal matrix of elements $-4(\pi s\Delta)^2$, $\mathbf{U}$ is the square matrix of elements $(2\pi/\lambda)^2\,\varepsilon(s\Delta-k\Delta)\Delta$, $\mathbf{E}$ is the vector-column of elements $E(s\Delta)$.

Thus, in the final case, the problem was led to the eigenvalue (square propagation constant) problem and the eigenvector (the discrete Fourier transform $E(x)$) problem which corresponds to preset value β. We can have few eigenvalues and appropriate eigenvectors which are orthogonal. By carrying out the inverse discrete Fourier transform of eigenvector we obtain field distribution $E(x)$.

In our numerical calculations, the results of which are shown below, we have used the simplest way to replace integral by sum according to second term of (10).

3. RESULTS

In this section analysis of planar waveguides by the method proposed was carried out for wavelength $\lambda = 1$ µm and thickness of waveguide middle layer $d = 2$ µm of permittivity $\varepsilon_1 = 2.25$. For symmetric waveguide $\varepsilon_0 = \varepsilon_2 = 1.96$. For asymmetric waveguide calculations were carried out at following values of permittivity: $\varepsilon_0 = 1$, $\varepsilon_1 = 2.25$, $\varepsilon_2 = 1.96$. Permittivity of gradient waveguide is described by Gaussian function, where $\varepsilon_0 = 1.96$, $\varepsilon_1 = 2.25$, $d = 2$ µm.

A. Symmetric planar waveguide for TE polarization waves. For the simplest symmetric planar waveguide permittivity can be written as:

$$\varepsilon(x)=\varepsilon_0+(\varepsilon_1-\varepsilon_0)\,\mathrm{rect}(x/d), \quad (12)$$

where function $\mathrm{rect}(x)=\begin{cases}1, |x|\le 1/2,\\ 0, |x|>1/2.\end{cases}$

The Fourier transform of (12) has a form [20]:

$$F\{\varepsilon(x)\}=\varepsilon_0\delta(u)+(\varepsilon_1-\varepsilon_0)\frac{\sin(\pi ud)}{\pi u}, \quad (13)$$

where $\delta(u)$ is Dirac delta function.

Taking into account filtering properties of Dirac delta function, one can write (8) in a form:

$$-4\pi^2u^2E(u)+\left(\frac{2\pi}{\lambda}\right)^2\varepsilon_0E(u)+\left(\frac{2\pi}{\lambda}\right)^2(\varepsilon_1-\varepsilon_0)\times$$

$$\times\int_{-\infty}^{\infty}\frac{\sin[\pi d(u-v)]}{\pi(u-v)}E(v)dv=\beta^2E(u). \quad (14)$$

Using a replacement integral by sum in (14), we can obtain equation in a discrete form (10) and, consequently, matrix equation to find eigenvalues and eigenvectors in a form (11). Propagation constants of localized modes β_j from the totality of eigenvalues of matrix equation must satisfy following correlation $2\pi\sqrt{\varepsilon_2}/\lambda < \beta_j < 2\pi\sqrt{\varepsilon_1}/\lambda$ [5]. For symmetric waveguide of above-mentioned data by solving dispersion equation [1, 5] next propagation constants were found: $\beta_1 = 9.347264$ µm^{-1}, $\beta_2 = 9.122738$ µm^{-1}, $\beta_3 = 8.818346$ µm^{-1}. All these decimal values are accurate. Propagation constants calculated by the method proposed for $N = 1001$ and $u_{max} = 40$ µm^{-1} are: $\beta_1 = 9.347264$ µm^{-1}, $\beta_2 = 9.122737$ µm^{-1}, $\beta_3 = 8.818345$ µm^{-1}. We see a good calculation accuracy of propagation constants by the method proposed. Propagation constants calculated according to the method described in [15] for $N = 1001$ and $x_{max} = 7.5$ µm are: $\beta_1 = 9.34730$ µm^{-1}, $\beta_2 = 9.12286$ µm^{-1}, $\beta_3 = 8.81654$ µm^{-1}. A value of $x_{max} = 7.5$ µm provides the best accuracy in this method at $N = 1001$. Apparently, to improve calculation accuracy of this method we have to significantly increase N and x_{max} that will inevitably lead to increasing of computation time. The results of numerical analysis of our method are shown in Fig. 3.

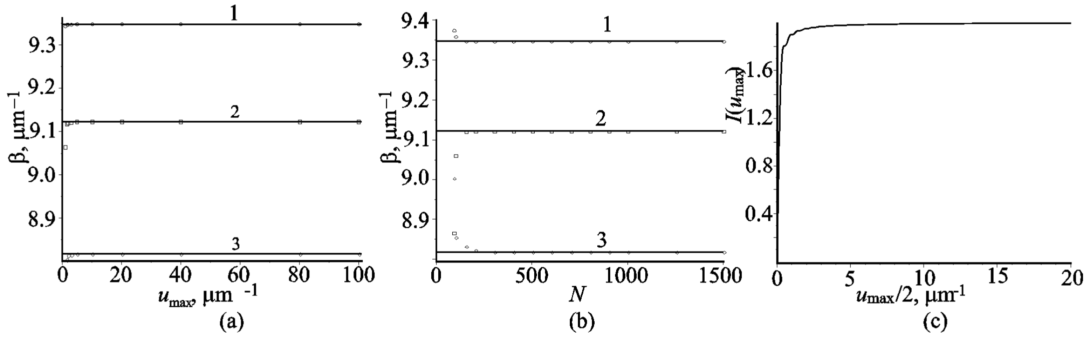

Fig. 3. Propagation constant dependence on u_{max} at $N = 1001$ (a); propagation constant dependence on N at $u_{max} = 40$ µm^{-1} (b); $I(u_{max})$ dependence on $u_{max}/2$, where $I(u_{max})=\int_{-u_{max}/2}^{u_{max}/2}\left[\frac{\sin(\pi ud)}{\pi u}\right]^2du$ (c). Points in Fig. 3, a and b, correspond to calculated values of propagation constants, horizontal lines to exact values of propagation constants

According to results of numerical analysis it is shown that reasonably accurate values of propagation constants at $N = 1001$ are obtained in a range of u_{max} from 10 μm^{-1} to 200 μm^{-1}. Bottom boundary of u_{max} can be explained by the fact that integral $I(u_{max})$ according to Fig. 3, c, is almost the maximum at $u_{max}/2 = 5$ μm^{-1}. Top boundary of u_{max} can be explained in the way that at oscillation half-period ($1/d$) of function $\sin(\pi ud)$ it should be no less than two samples, i. e., $1/d = 0.5$ μm^{-1} $> 2\Delta = 2u_{max}/N = 2\times200/1001 \approx 0.4$ μm^{-1}, that is agreed with Fig. 3, a. According to Fig. 3, b, inaccuracy of analysis significantly increases at $N < 200$. In other words, following ratio should be implemented again: $1/d = 0.5$ μm^{-1} $> 2\Delta = 2u_{max}/N = 2\times40/201 \approx 0.4$ μm^{-1}.

B. Asymmetric planar waveguide for TE polarization waves. Usage of the method directly for numerical analysis of asymmetric waveguide does not provide a satisfactory accuracy to determine propagation constants, as in functional dependence of permittivity such component appears, which is proportional to function sgn(x). It causes the Hilbert transform in integral equation, i. e., in integral singular kernel such as $1/(u-v)$ exists. Therefore, asymmetric waveguide must be modified, as it is shown by dashed lines in Fig. 1, b. Thus, for modified waveguide $\varepsilon(-\infty) = \varepsilon(\infty) = \varepsilon_2$. Then the question arises: what value would b have? Probably, b would be such, that field of localized mode at $x = -b$ would be much smaller than at $x = 0$, i. e., its value can be equated to zero. However, b would not be too large, because such a waveguide will approach to non-modified. The next analysis will give answer to this question.

Permittivity of modified waveguide can be written as:

$$\varepsilon(x) = \varepsilon_2 + (\varepsilon_1 - \varepsilon_2)\mathrm{rect}\left(\frac{x}{d}\right) - $$
$$-(\varepsilon_2 - \varepsilon_0)\mathrm{rect}\left[\frac{1}{b - d/2}\left(x + \frac{b + d/2}{2}\right)\right]. \quad (15)$$

The Fourier transform of (15) has a form [20]:

$$F\{\varepsilon(x)\} = \varepsilon_2\delta(u) + (\varepsilon_1 - \varepsilon_2)\frac{\sin(\pi ud)}{\pi u} - $$
$$-(\varepsilon_2 - \varepsilon_0)\frac{\sin[\pi u(b - d/2)]}{\pi u}\exp[i\pi u(b + d/2)]. \quad (16)$$

Further, we act such as in the case of symmetric waveguide, i. e., we substitute (16) into (8) and replace integral by sum. Exact values of propagation constants are $\beta_1 = 9.336874$ μm^{-1} and $\beta_2 = 9.077763$ μm^{-1}. Numerical values of propagation constants obtained by the method proposed at $N = 1001$, $u_{max} = 40$ μm^{-1} and $b = 10$ μm are $\beta_1 = 9.3368736$ μm^{-1} and $\beta_2 = 9.0777634$ μm^{-1}. We see a good match of numerical values of propagation constants obtained by two methods. The results of analysis for asymmetric modified waveguide are shown in Fig. 4.

We see that at $N = 1001$ and $u_{max} = 40$ μm^{-1} according to Fig. 4, b, value of b can vary widely from 2 μm to 23 μm. Outside these boundaries an error in calculation of propagation constants increases dramatically. Bottom boundary of b is consistent with Fig. 4, c, i. e., at $x < -2$ field is almost zero. That is, bottom boundary of b should be selected from condition $b >> d/2$. It is impractical to take too big value of b, because it requires taking big value of N at numerical calculations that leads to increasing of analysis time. It is seen in Fig. 4, a, under which error increases dramatically at $N < 500$. According to Fig. 3, a, for boundary value $N < 500$, $u_{max} = 40$ μm^{-1} and $b = 10$ μm or $u_{max}b/N = 0.8$. According to Fig. 3, b, values of $N = 1001$ and $u_{max} = 40$ μm^{-1} correspond to boundary value $b = 23$ μm or $u_{max}b/N = 0.86$. That is, top boundary of b can be selected from condition $b < 0.8N/u_{max}$. In general, a choice of b can be defined as:

$$d/2 << b < 0.8N/u_{max}, \quad (17)$$

i. e., in this example value of $b = 10$ μm is quite acceptable.

Propagation constants calculated according to finite difference method [15, 16] at $N = 1001$ and $x_{max} = 7.5$ μm are $\beta_1 = 9.33692$ μm^{-1} and $\beta_2 = 9.07792$ μm^{-1}, i. e., propagation constants are calculated with less accuracy than in our method.

C. Symmetric planar gradient waveguide for TE polarization waves. For example, let's consider waveguide, which permittivity is described by Gaussian distribution:

$$\varepsilon(x) = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\exp\left[-\pi(x/d)^2\right]. \quad (18)$$

The Fourier transform of this function is [20]:

$$F\{\varepsilon(x)\} = \varepsilon_0\delta(u) + d(\varepsilon_1 - \varepsilon_0)\exp\left[-\pi(du)^2\right]. \quad (19)$$

Further, we substitute (19) into (8) and replace integration by summation. Fig. 5 shows the results of analysis of gradient symmetric waveguide. The results of calculation of propagation constants depending on u_{max} at $N = 1001$ are given in Table 1.

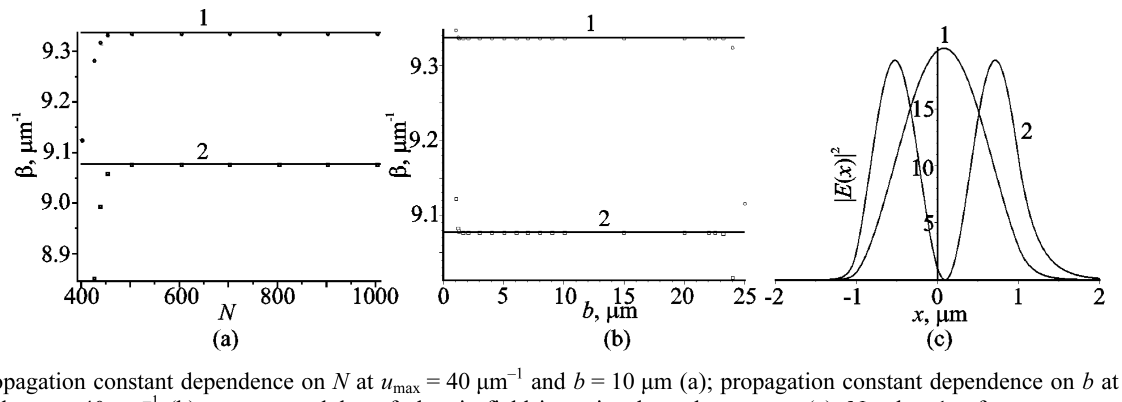

Fig. 4. Propagation constant dependence on N at $u_{max} = 40$ μm^{-1} and $b = 10$ μm (a); propagation constant dependence on b at $N = 1001$ and $u_{max} = 40$ μm^{-1} (b); square modulus of electric field intensity dependence on x (c). Number 1 refers to waveguide mode $\beta_1 = 9.3368736$ μm^{-1}, number 2 to $\beta_2 = 9.0777634$ μm^{-1}. In Fig. 3, a, b, horizontal lines correspond to exact values of propagation constants, points to calculated ones by the method proposed

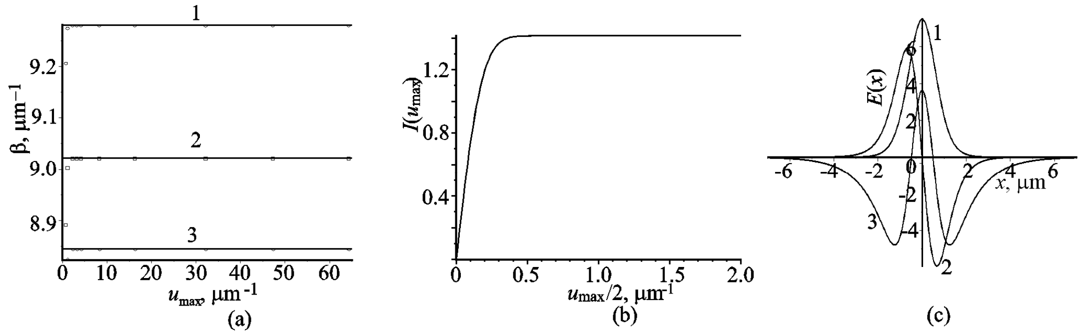

Fig. 5. Propagation constant dependence on $u_{\max}$ at $N = 1001$ (a); $I(u_{\max})$ dependence on $u_{\max}/2$, where $I(u_{\max}) = \int_{-u_{\max}/2}^{u_{\max}/2} d^2 \exp\left[-2\pi(du)^2\right] du$ (b); electric field intensity dependence on x (c)

Table 1. Propagation constant dependence on maximum spatial frequency

Maximum spatial frequency, μm^{-1}	Propagation constant β_1, μm^{-1}	Propagation constant β_2, μm^{-1}	Propagation constant β_3, μm^{-1}
2	9.28031898	9.02219297	8.84490965
3	9.28032187	9.02220908	8.84493647
4	9.28032187	9.02220908	8.84493647
32	9.28032187	9.02220908	8.84493647
47	9.28032187	9.02220908	8.84493647
64	9.28032187	9.02220908	8.84493722

A peculiarity of Table 1 and Fig. 5, a, is that when $u_{\max}$ is changing from 3 μm^{-1} to 47 μm^{-1} calculated propagation constants are unchanged, and that's why has values: $\beta_1 = 9.28032187\ \mu m^{-1}$, $\beta_2 = 9.02220908\ \mu m^{-1}$ and $\beta_3 = 8.84493647\ \mu m^{-1}$. Therefore, these values can be considered as accurate ones. Bottom boundary of $u_{\max}$ can be explained by the fact that integral $I(u_{\max})$ in Fig. 5, b, is almost the maximum at $u_{\max} = 1\ \mu m^{-1}$. Significant deviations from exact values appear only at $u_{\max} = 200\ \mu m^{-1}$, and they are only for β_3. Propagation constants calculated by the finite difference method [15] at $x_{\max} = 7.5\ \mu m$ are $\beta_1 = 9.280323\ \mu m^{-1}$, $\beta_2 = 9.022207\ \mu m^{-1}$, $\beta_3 = 8.843480\ \mu m^{-1}$, and at $x_{\max} = 14\ \mu m$ they are $\beta_1 = 9.280326\ \mu m^{-1}$, $\beta_2 = 9.022224\ \mu m^{-1}$ and $\beta_3 = 8.844950\ \mu m^{-1}$.

D. Symmetric planar waveguide for TM polarization waves. Previous examples concern planar waveguides where TE polarization waves propagate. These examples feature such values of permittivity $\varepsilon(x)$, for which it is easy to find the Fourier transform analytically. At the same time, for gradient waveguides it is difficult to do it for expression $\varepsilon'(x)/\varepsilon(x)$ (see equation 3), that is necessary for TM polarization waves. However, it can be done for symmetric waveguide according to Fig. 1, a. Therefore, we can express $\varepsilon'(x)/\varepsilon(x)$ as:

$$\frac{1}{\varepsilon(x)}\frac{d\varepsilon(x)}{dx} = \frac{1}{2}\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_0}\right)\cdot(\varepsilon_1-\varepsilon_0)\times \left[\delta\left(x+\frac{d}{2}\right)-\delta\left(x-\frac{d}{2}\right)\right]. \quad (20)$$

Carrying out the Fourier transform of (20), we obtain:

$$F\left\{\frac{1}{\varepsilon(x)}\frac{d\varepsilon(x)}{dx}\right\} = -i\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_0}\right)\cdot(\varepsilon_1-\varepsilon_0)\cdot\sin(\pi u d). \quad (21)$$

Thus, equation (9) for given permittivity dependence on x has a form:

$$-4\pi^2u^2H(u)+2\pi(\varepsilon_1-\varepsilon_0)\cdot\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_0}\right)\times \int_{-\infty}^{\infty}\sin[\pi d(u-v)]vH(v)dv + \left(\frac{2\pi}{\lambda}\right)^2\int_{-\infty}^{\infty}\varepsilon(u-v)H(v)dv = \beta^2H(u). \quad (22)$$

According to calculations three localized modes can propagate in this waveguide. Exact values of propagation constants obtained by solving the dispersion equation [1, 5] are: $\beta_1 = 9.3428215\ \mu m^{-1}$, $\beta_2 = 9.1099993\ \mu m^{-1}$, $\beta_3 = 8.8148978\ \mu m^{-1}$. At computing process parameters $N = 1001$ and $u_{\max} = 60\ \mu m^{-1}$ following propagation constants are: $\beta_1 = 9.342825\ \mu m^{-1}$, $\beta_2 = 9.110037\ \mu m^{-1}$, $\beta_3 = 8.814998\ \mu m^{-1}$. We can see a satisfactory accuracy of calculations, but it is worse than regarding TE polarization waves. This can be explained by the fact that the Fourier transform of $\varepsilon'(x)/\varepsilon(x)$ doesn't decrease to zero (due to first type discontinuity of function $\varepsilon(x)$) at $u \to \pm\infty$. Due to such discontinuity of function $\varepsilon(x)$ it is impossible to determine propagation constants of this waveguide by the finite difference method described in [15, 16].

However, apparently, we can expect that for continuous functions $\varepsilon(x)$ calculation accuracy of propagation constants will increase significantly. It is confirmed by next example:

$$\varepsilon(x) = \begin{cases} \varepsilon_0, |x| > d/2, \\ \varepsilon_1 - (\varepsilon_1-\varepsilon_0)\dfrac{x^2}{(d/2)^2}, |x| \le d/2. \end{cases} \quad (23)$$

Permittivity according to (23) is a continuous function, but its derivative has first type discontinuity. As a result, the Fourier transform of $\varepsilon'(x)/\varepsilon(x)$ is a continuous function which has oscillations, and it tends to zero at $x \to \pm\infty$. This waveguide has two propagation constants. When maximum spatial frequency $u_{\max}$ is changing from 20 μm^{-1} to 60 μm^{-1} propagation constant $\beta_1 = 9.2390569\ \mu m^{-1}$, and when it is doing from 10 μm^{-1} to 35 μm^{-1} propagation constant $\beta_2 = 8.9028720$ μm^{-1}. Calculation of these propagation constants was held at $N = 401$. We can see high calculation accuracy by the method proposed.

4. CONCLUSIONS

A new numerical method to determine propagation constants of gradient planar waveguide modes and their appropriate fields is developed. The method is based on fact that for planar waveguide modes appropriate functions describing fields are completely integrated. Hence, the Fourier transform can be applied to wave equations to enable the move from differential wave equations to integral ones. Finally, we obtain the eigenvalue/ eigenvector problem where eigenvalues are square propagation constants, and corresponding eigenvectors are discrete field Fourier transforms of waveguide modes. Using the inverse Fourier transform we obtain field distribution for appropriate waveguide modes.

At sufficiently large values of N (> 500) propagation constants don't practically change while frequency varies within certain limits (Table 1). It can be explained by fact that not only a field tends rapidly to zero at $x \to \pm\infty$, but also the Fourier transform do it rapidly at $u \to \pm\infty$. Therefore, at right choice of values of N and $u_{\max}$ the Whittaker-Shannon sampling theorem is performed of high accuracy. This method is characterized by good numerical stability.

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